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Compressible flow past a contour and stationary vortices

By A. BARSONY-NAGY, J. ER-EL AND S. YUNGSTER
Department of Aeronautical Engineering, Technion – Israel Institute of Technology, Haifa 32000, Israel

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The Rayleigh–Janzen expansion method is extended to plane and steady flows which contain one or more point vortices interacting with a smooth or sharp-edged obstacle. A uniformly valid approximate solution of the compressible-flow equations is deduced by applying a perturbation method and by using matched asymptotic expansions to solve the resulting singular perturbation problem. The method yields compressibility corrections for the vortex positions and for the velocities. Results are presented for the flow past a circle and a pair of symmetric vortices (Föppl's flow). They show that the compressibility effects are substantial and are consistent with experimental data.

1. Introduction

Several important aerodynamic problems involve the calculation of two-dimensional inviscid flows past impermeable contours and concentrated vortices. These problems include the modelling of the cross-flow past slender bodies at high angles of attack (Bryson 1959; Nielsen 1960) and of the flow about airfoils with trapped vortices (Saffman & Sheffield 1977, Huang & Chow 1982). For such flows, even at relatively low subsonic free-stream velocities, the compressibility effects can be significant. However, closed-form solutions have been obtained for the incompressible case only. The extension of these solutions to the compressible range is of theoretical as well as practical interest.

A classical method for evaluating compressibility corrections for subsonic flows is the well-known Rayleigh–Janzen expansion. When the flows contain point vortices this expansion fails near the vortex centres because compressibility effects are large there. The objective of the present paper is to extend the Rayleigh–Janzen expansion to the steady and plane flow past a contour and stationary vortices placed in a subsonic stream. For such flows the Rayleigh–Janzen expansion can be regarded as an outer solution, valid away from any vortex, which will be matched with an inner solution that holds near the centre of each vortex. A similar procedure was recently used by Moore (1985) in his study of the vortex ring in a compressible fluid. However, Moore's considerations cannot be applied directly to flows past obstacles, mainly because in his case the compressibility effects in the outer region are negligible.

The general solution is illustrated with an example of the flow past a circle and a symmetric pair of vortices placed in a subsonic stream. For this relatively simple flow, known in the incompressible case as the Föppl (1913) flow, the compressibility corrections are obtained explicitly. The results presented show the effects of compressibility on the equilibrium positions of the vortices, on the velocities and on the streamlines.

2. Solution in the region of subsonic flow

First we consider the inviscid, steady and plane flow past a closed contour and a single stationary vortex placed in a subsonic stream. The x-axis is parallel to the velocity U of the free stream and the origin of the coordinates is taken inside the contour. We assume that the velocities at the contour are subsonic, so that the vicinity of the vortex centre where the flow is supersonic is completely surrounded by subsonic flow.

In the region in which the velocities are subsonic, an approximation of the compressible flow is obtained by using the Imai–Lamla method (Jacob 1959)†. This method extends the concept of complex velocity potential to compressible, plane and irrotational flows. In applying the method, the coordinates x and y are changed into the non-dimensional complex variables

$$z = \frac{x+\mathrm{i}y}{l}; \quad \bar{z} = \frac{x-\mathrm{i}y}{l}, \tag{1}$$

where l is a characteristic length of the contour. Then for the plane and irrotational flow the dimensionless complex velocity potential is introduced as

$$f(z,\bar{z}) = \frac{1}{Ul}[\phi(x,y)+\mathrm{i}\psi(x,y)]. \tag{2}$$

Here ϕ is the real velocity potential and ψ is the stream function. The compressible Euler equations for an inviscid and perfect gas are written in terms of the complex velocity potential and successive approximations of the solution are obtained by expanding $f(z,\bar{z})$ asymptotically with respect to the parameter

$$M_0 = \frac{U}{a_0} < 1, \tag{3}$$

where a_0 denotes the speed of the sound in the fluid at rest. Retaining only the first two terms of this expansion we have

$$f(z,\bar{z}) = f_0(z) + M_0^2 f_1(z,\bar{z}) + o(M_0^2). \tag{4}$$

The leading term $f_0(z)$ in the expansion of $f(z,\bar{z})$ is an analytic function of the complex variable z only and it represents the complex potential of an incompressible flow past the considered configuration. In this incompressible flow the velocity U of the free stream and the circulation around the vortex are the same as in the compressible flow and, if the contour has a sharp trailing edge, the Kutta condition is satisfied, i.e. the velocity at the edge is finite.

The second term in the expansion of the complex velocity potential is given by (Jacob 1959)

$$f_1(z,\bar{z}) = \frac{1}{4}\frac{\mathrm{d}f_0}{\mathrm{d}z}\overline{\int_{z_1}^{z}\left(\frac{\mathrm{d}f_0}{\mathrm{d}z}\right)^2 \mathrm{d}z} + \tfrac{1}{4}g(z), \tag{5}$$

where z_1 is the coordinate of the sharp trailing edge or, when there is no such edge, of an arbitrary point in the flow field and $g(z)$ is a function that has to be determined from the following requirements:

(i) The imaginary part of $f_1(z,\bar{z})$ is a uniform single-valued function in the flow field.

† The Imai–Lamla method is a variant, suited particularly to plane flows, of the Rayleigh–Janzen expansion (see Jacob 1959).

(ii) The contour is a streamline in the second (i.e. of order M_0^2) approximation of the flow. Hence the imaginary part of $f_1(z,\bar{z})$ is constant at the contour.

(iii) The contributions of $f_1(z,\bar{z})$ to the velocity of the free stream and to the circulation around the vortex are zero.

(iv) If the contour has a sharp trailing edge, the velocity is finite at this edge.

(v) The velocity has a vortex-type singularity when it approaches the vortex centre, i.e. it becomes infinite as the inverse of the distance to the centre.

Requirement (i) expresses mass conservation, (ii) and (iii) are the boundary conditions for the second approximation, and (iv) is the Kutta condition. The last requirement (v) is really a matching condition that could have been postponed to the discussion of the inner solution near the vortex centre. We prefer, however, to invoke this requirement now, because it enables us to evaluate completely the outer solution prior to, and independent of, considerations regarding the inner solution.

In determining $f_1(z,\bar{z})$ that satisfies the above conditions, we have to take into account that the integral in (5) is a non-uniform, multi-valued function in the flow field. This behaviour is due to the singularities of $\mathrm{d}f_0/\mathrm{d}z$ at $z=\infty$ and at the vortex position $z=\zeta$. The non-uniform parts of the integral can be evaluated by expanding the integrand $(\mathrm{d}f_0/\mathrm{d}z)^2$ in power series near its singularities. The resulting expressions involve the strength of the vortex Γ and the circulation Γ_0 around the contour in the incompressible approximation of the flow. Denoting

$$k=\frac{\Gamma}{2\pi Ul}, \tag{6}$$

$$k_0=\frac{\Gamma_0}{2\pi Ul}, \tag{7}$$

we obtain

$$\int_{z_1}^{z}\left(\frac{\mathrm{d}f_0}{\mathrm{d}z}\right)^2\mathrm{d}z = I(z)+2\mathrm{i}(k+k_0)\ln\frac{z}{z_1}+2\mathrm{i}kC_0(\zeta)\ln\left(\frac{z-\zeta}{z_1-\zeta}\frac{z_1}{z}\right), \tag{8}$$

where $I(z)$ is a single-valued function in the flow field and $C_0(\zeta)$ is the free term in the following expansion of $\mathrm{d}f_0/\mathrm{d}z$ near the vortex centre:

$$\frac{\mathrm{d}f_0}{\mathrm{d}z}=-\frac{ik}{z-\zeta}+\sum_{n=0}^{\infty}C_n(\zeta)(z-\zeta)^n. \tag{9}$$

Now, the evaluation of $f_1(z,\bar{z})$ can be simplified by taking into account that the vortex is stationary. In the incompressible approximation of the flow the coordinate ζ_0 of the centre of the stationary vortex is the solution of the equation (Milne-Thomson 1968)

$$C_0(\zeta_0)=0. \tag{10}$$

In the compressible flow the vortex centre is shifted to $z=\zeta$, but the difference $\zeta-\zeta_0$ and, consequently, $C_0(\zeta)$ vanishes when M_0 approaches zero. Therefore, the term involving $C_0(\zeta)$ in the expression of the integral in (8) can be neglected, since, in view of (4) and (5), its contribution to the complex velocity potential is of the order of magnitude $C_0(\zeta)M_0^2=o(M_0^2)$. Then, the function $f_1(z,\bar{z})$ satisfies the requirements (i), (iii) and (v) if $g(z)$ in (5) has the form

$$g(z)=g_1(z)-z-2\mathrm{i}(k+k_0)\frac{\mathrm{d}f_0}{\mathrm{d}z}\ln\frac{z}{z_1}-\mathrm{i}k_1\ln z+\frac{D(\zeta)}{z-\zeta}. \tag{11}$$

Here g_1 is a single-valued analytic function bounded outside the contour, k_1 is a real constant related to the additional circulation around the contour due to the effects of compressibility, and

$$D(\zeta) = \mathrm{i}\lim_{z\to\zeta}\left\{\left[\overline{\frac{\mathrm{d}}{\mathrm{d}z}((z-\zeta)\,I(z))}\right]-4(k+k_0)\ln\left|\frac{\zeta}{z_1}\right|\right\}. \tag{12}$$

Combining (5), (8) and (11) we have

$$f_1(z,\bar{z}) = \tfrac{1}{4}[N(z,\bar{z})+g_1(z)], \tag{13}$$

where

$$N(z,\bar{z}) = \frac{\mathrm{d}f_0}{\mathrm{d}z}\left[\overline{I(z)}-4\mathrm{i}(k+k_0)\ln\left|\frac{z}{z_1}\right|\right]-z-\mathrm{i}k_1\ln z+\frac{D(\zeta)}{z-\zeta}. \tag{14}$$

In terms of the function $g_1(z)$, the conditions (ii) and (iv) satisfied by $f_1(z,\bar{z})$ become

$$\mathrm{Im}\,(g_1(z)) = -\mathrm{Im}\,(N(z,\bar{z}))+\text{const} \quad \text{for } |z| = 1, \tag{15}$$

$$\lim_{z\to z_1}\left|\frac{\mathrm{d}g_1(z)}{\mathrm{d}z}\right| < \infty. \tag{16}$$

Once the analytic function $g_1(z)$ and the constant k_1 are determined, the second approximation of the complex velocity potential is obtained. An apparent difficulty in the calculations is that $N(z,\bar{z})$ and, therefore the boundary values of $g_1(z)$ at the contour, depend on the unknown coordinate ζ of the centre of the stationary vortex. However, in evaluating $g_1(z)$ and k_1 we can assume that the vortex is situated at its incompressible equilibrium position $z = \zeta_0$, since, owing to this assumption, the resulting error in the complex velocity potential is of $o(M_0^2)$. Then, the calculation of $g_1(z)$ and k_1 becomes a standard problem from the theory of analytic complex functions. This problem is discussed in detail by Jacob (1959), who solved it using conformal mapping for the exterior of the contour on the exterior of a circle.

To complete the evaluation of the approximate complex velocity potential the stationary position of the vortex in the compressible flow has to be determined. This position is obtained from the condition that the force acting on the vortex is zero. The force is evaluated by using an extension of Blasius' theorem to compressible flow (Barsony-Nagy 1985). Then, requiring that the significant terms of the force (i.e. the terms of order of magnitude larger than or equal to M_0^2) vanish, we get the following expansion for ζ:

$$\zeta = \zeta_0+M_0^2\,\zeta_1+o(M_0^2). \tag{17}$$

Here ζ_0 is the solution of (10) and

$$\zeta_1 = \frac{\mathrm{i}}{4}\left(\frac{\mathrm{d}C_0}{\mathrm{d}\zeta}(\zeta_0)\right)^{-1}\left[\frac{\mathrm{d}g_1}{\mathrm{d}z}(\zeta_0)+\frac{k(k+k_0)}{\zeta_0^2}-1-\frac{\mathrm{i}k_1}{\zeta_0}-\frac{\mathrm{i}}{k}D(\zeta_0)\,C_1(\zeta_0)\right], \tag{18}$$

where the coefficients $C_0(\zeta), C_1(\zeta)$ and $D(\zeta)$ are defined in (9) and (12). In this way the first-order compressibility correction for the position of the stationary vortex is expressed in terms of the vortex position in the incompressible approximation of the flow.

In conclusion, the evaluation of the second approximation of the compressible flow involves the following steps. First, the analytic function $g_1(z)$ and the constants k_1 and ζ_1 are calculated by assuming that the vortex is situated at its equilibrium position ζ_0 in incompressible flow. Then, the centre of the vortex is moved to ζ as indicated in (17) and $f_0(z)$ and $N(z,\bar{z})$ are recalculated for new positions of the vortex.

Finally, the second approximation of the complex velocity potential is obtained by combining (4) and (13).

We note that the method developed to calculate the approximate solution when there is one vortex in the flow field can be extended readily to the case of any finite number of vortices. In this case, requirements (i)–(iv) remain unchanged, whereas requirement (v) (regarding the singularity of the solution) has to be fulfilled for each vortex. In addition, we assume that the regions of supersonic flow around the centres of the vortices do not intersect. Then, the extension of the solution to n vortices involves the modification of the functions $I(z)$ and $N(z,\bar{z})$, only. Denoting by $\zeta^{(j)}$ and $\Gamma^{(j)}$ $(j = 1, 2, \ldots, n)$ the coordinates of the vortices and their strengths respectively, and introducing the constants $k^{(j)} = \Gamma^{(j)}/2\pi Ul$ $(j = 1, 2, \ldots, n)$, (8) and (14) are replaced by

$$\int_{z_1}^{z} \left(\frac{\mathrm{d}f_0}{\mathrm{d}z}\right)^2 \mathrm{d}z = I(z) + 2\mathrm{i}\left(k_0 + \sum_{j=1}^{n} k^{(j)}\right) \ln\frac{z}{z_1} + o(1) \tag{19}$$

and

$$N(z,\bar{z}) = \frac{\mathrm{d}f_0}{\mathrm{d}z}\left[I(z) - 4\mathrm{i}\left(k_0 + \sum_{j=1}^{n} k^{(j)} \ln\frac{z}{z_1}\right)\right] - z - \mathrm{i}k_1 \ln z + \sum_{j=1}^{n} \frac{D^{(j)}(\zeta^{(j)})}{z - \zeta^{(j)}}, \tag{20}$$

where the coefficients $D^{(j)}$ $(j = 1, 2, \ldots, n)$ are

$$D^{(j)} = \mathrm{i}k^{(j)} \lim_{z\to\zeta^{(j)}} \overline{\left\{\frac{\mathrm{d}}{\mathrm{d}z}[(z-\zeta^{(j)})\, I(z)] - 4\left(k_0 + \sum_{m=1}^{n} k^{(m)}\right) \ln\frac{z}{z_1}\right\}}. \tag{21}$$

Finally, the stationary positions of the vortices are found by solving a set of simultaneous equations obtained from the condition that the force acting on each one of the vortices vanishes.

3. Solution in the vicinity of the vortex centre

The approximate solution deduced in the previous section breaks down when $z - \zeta^{(j)} = O(k^{(j)}M_0)$, as the first two terms in the expansion of the complex velocity potential become of the same order of magnitude. A uniformly valid approximation of the compressible flow will be obtained by supplementing and matching this (outer) solution with a local (inner) solution that holds near the centre of the vortex. To calculate the inner solution it is sufficient to consider the case when there is a single vortex in the flow and it is convenient to work with the real velocity potential ϕ. We introduce the polar coordinates r and θ by putting

$$z - \zeta = r\,\mathrm{e}^{\mathrm{i}\theta}. \tag{22}$$

Then, the approximate outer solution fails for $r = O(kM_0)$ and, as is usually done in the method of matched asymptotic expansions (see Van Dyke 1964), a 'stretched' inner coordinate is defined. In our case the appropriate inner coordinate r_1 is

$$r_1 = \frac{r}{kM_0}. \tag{23}$$

We start by determining the asymptotic behaviour of the outer solution which is required in the matching process. For this purpose, the velocity potential ϕ (obtained by taking the real part of the second approximation of the complex velocity potential deduced in the previous section) is expressed as a function of the variables r_1 and

θ and the result is expanded for small M_0 with r_1 fixed. Neglecting the terms of order higher than M_0^2 and returning to the variable r we have

$$\frac{\phi(r_1,\theta)}{lU} \sim k\theta + (r^2 + \tfrac{3}{2}k^2M_0^2)\,\mathrm{Re}\,(C_1\,\mathrm{e}^{2\mathrm{i}\theta}) \quad \text{for } r = O(kM_0), \tag{24}$$

where C_1 is the coefficient of $(z-\zeta)$ in the expansion of $\mathrm{d}f_0/\mathrm{d}z$ near the centre of the vortex as indicated in (9).

The inner solution $\Phi(r_1,\theta)$ is evaluated by using asymptotic expansions with respect to M_0 for r_1 fixed. Equation (24) suggests that the expansion of $\Phi(r_1,\theta)$ has the form

$$\frac{\Phi(r_1,\theta)}{lU} = k\theta + \Delta(M_0)\,\Phi_1(r_1,\theta) + o(M_0^2). \tag{25}$$

Here the leading term in the expansion is the exact solution for an isolated compressible vortex (Jacob 1959) and $\Delta(M_0) < 1$ is a gauge function that has to be determined from the matching conditions. Introducing $\Phi(r_1,\theta)$ into the well-known equation of the velocity potential in compressible flow (Lighthill 1955), we obtain that $\Phi_1(r_1,\theta)$ satisfies the linear differential equation

$$\left(1-\frac{\gamma-1}{2r_1^2}\right)\frac{\partial^2\Phi_1}{\partial r_1^2} + \frac{1}{r_1}\left(1-\frac{\gamma-3}{2r_1^2}\right)\frac{\partial\Phi_1}{\partial r_1} + \frac{1}{r_1^2}\left(1-\frac{\gamma+1}{2r_1^2}\right)\frac{\partial^2\Phi_1}{\partial\theta^2} = 0, \tag{26}$$

where γ is the ratio of the specific heats. In (26) the coefficient of the derivative $\partial^2\Phi_1/\partial r_1^2$ vanishes at the circle $r_1 = [\frac{1}{2}(\gamma-1)]^{\frac{1}{2}}$ where the velocity generated by the isolated vortex reaches its highest possible value in isentropic flow. Inside this circle the potential solution has no physical meaning. When r_1 increases, (26) changes its type from hyperbolic to elliptic at the circle $r_1 = [\frac{1}{2}(\gamma+1)]^{\frac{1}{2}}$, which corresponds to the sonic line for the isolated vortex. We assume that there are no shock waves in the flow, so that $\Phi_1(r_1,\theta)$ and its derivatives are continuous. This assumption is reasonable as long as the region of supersonic flow does not reach the contour. Then, it can be shown that (25) has a unique solution that can be matched with the outer solutions and is bounded at $r = [\frac{1}{2}(\gamma-1)]^{\frac{1}{2}}$. Fortunately, this solution can be obtained by using the method of separation of the variables, and we have

$$\Phi_1(r_1,\theta) = Ar_1^2\,F\left(a,b;c\,\middle|\left(1-\frac{\gamma-1}{2r_1^2}\right)\right)\mathrm{Re}\,(C_1\,\mathrm{e}^{2\mathrm{i}\theta}), \tag{27}$$

where A is a constant, $F(a,b;c\,|\,s)$ is the hypergeometric function and we wrote for brevity

$$a = \frac{3-2\gamma+(4\gamma^2-3)^{\frac{1}{2}}}{2(\gamma-1)}, \quad b = \frac{3-2\gamma-(4\gamma^2-3)^{\frac{1}{2}}}{2(\gamma-1)}, \quad c = a+b+2 = \frac{1}{\gamma-1}. \tag{28}$$

The inner solution $\Phi(r_1,\theta)$ is obtained by combining (25) and (27). To match it with the outer solution, Φ is expressed as a function of r and then expanded asymptotically with respect to M_0 for fixed r. The terms of order lower than or equal to M_0^2 in the resulting expansion can be evaluated readily by using a connection formula for the hypergeometric function (Abramowitz & Stegun 1965). These terms coincide with those on the right-hand side of (24) if

$$\Delta(M_0) = M_0^2, \tag{29}$$

$$A = k^2\frac{\Gamma(a+2)\,\Gamma(b+2)}{\Gamma(c)}, \tag{30}$$

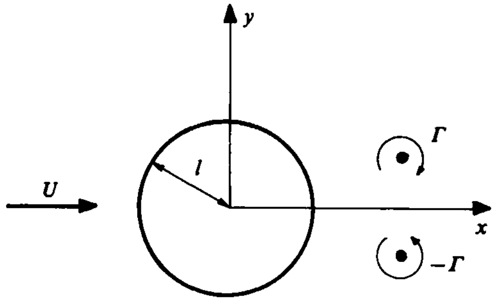

FIGURE 1. Geometry of Föppl's flow.

where $\Gamma(s)$ is Euler's gamma-function. Finally, introducing into (25) $\Phi_1, \Delta(M_0)$ and A given by (27), (29) and (30) respectively, the inner solution becomes

$$\frac{\Phi(r,\theta)}{Ul} = k\theta + \frac{\Gamma(a+2)\,\Gamma(b+2)}{\Gamma(c)} r^2 F\left(a, b; c \left| \left(1 - \frac{\gamma-1}{2r^2} k^2 M_0^2\right)\right.\right) \mathrm{Re}\,(C\,\mathrm{e}^{2\mathrm{i}\theta}) + o(M_0^2) \quad \text{for } \frac{r}{l} = O(kM_0). \tag{31}$$

We have deduced an approximate inner solution that describes the flow in the vicinity of the centre of any stationary vortex completely surrounded by subsonic flow. When there are several vortices in the flow field, the inner solution holds for each one separately. In this case the constants k and C_1 are replaced by the appropriate values for each vortex. It should be noted that, in our approach based on potential theory, a region of vacuum is formed close to the centre of the vortex. A more realistic model that avoids the region of vacuum can be obtained by introducing a rotational vortex core. However, this model is not discussed here, since it involves considerations related to rotational and non-isentropic flows which are beyond the scope of the present paper.

We would like to mention, also, that an equation similar to (26) was deduced by Moore (1985) in his study of a compressible vortex ring. Although Moore's equation is somewhat more general (it includes a free term due to the curvature of the vortex), the velocity potential Φ given by (31) cannot be obtained from his solution, since in his case the compressibility effects are negligible far away from the vortex ring.

4. An example

As an application, we extend the classical Föppl solution (Föppl 1913) to the compressible range. We consider the flow past a circle and a pair of symmetric vortices placed in a subsonic stream (figure 1). This configuration can be regarded as an approximate model of the cross-flow of an infinite yawed circular cylinder (Bryson 1959).

The complex velocity potential $f_0(z)$ of the incompressible flow past the configuration considered and the corresponding stationary positions of the vortices are well known (see Milne-Thomson 1968). In the present case the flow symmetry simplifies essentially the calculation of the compressibility corrections to the outer solution. For this flow the Kutta condition is not needed and it is replaced by the requirement that there is no circulation around the circle. Then, since the sum of the circulations around the vortices is also zero, all the logarithmic terms drop out from the expression

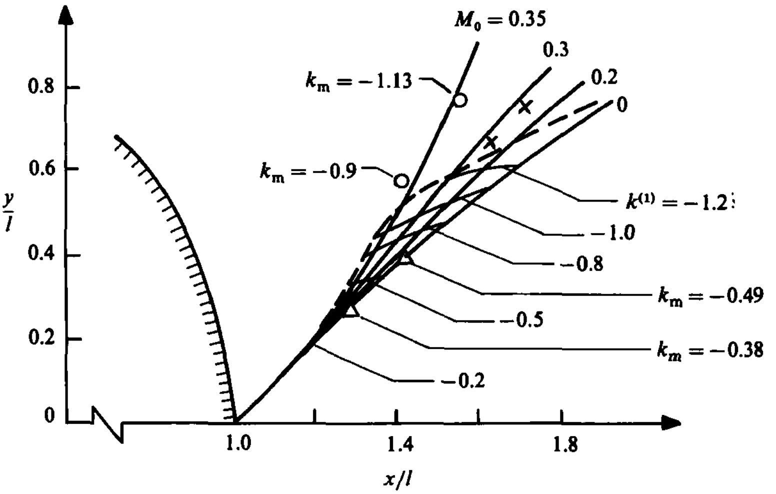

FIGURE 2. Equilibrium positions of the vortices. Present theory: ——, computed vortex positions; ---, validity boundary of the solution. Experiments: △, $M_0 = 0$ (Fidler *et al.* 1977); ×, $M_0 = 0.28$ (Jorgensen 1977 – in Jorgensen's study no vortex strengths are given); ○, $M_0 = 0.353$ (Owen & Johnson 1979).

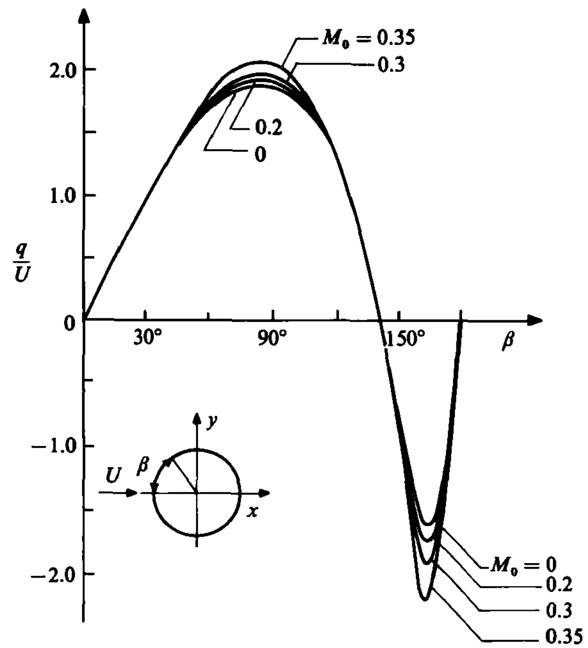

FIGURE 3. Effects of compressibility on the velocities at the circle ($k^{(1)} = -0.8$).

of the function $N(z, \bar{z})$ given in (20). Choosing the radius of the circle as the characteristic length l in the definition of the dimensionless variables, the boundary condition satisfied by $g_1(z)$ at the circle, (15), can be written in the form

$$\operatorname{Im}(g_1(z)) = \operatorname{Im}\overline{(N(\bar{z}^{-1}, \bar{z})} + \text{const} \quad \text{for } |z| = 1. \tag{32}$$

According to Schwarz's symmetry principle and the circle theorem (Milne-Thomson 1968), $\overline{N(\bar{z}^{-1}, \bar{z})}$ is an analytic function of the variable z alone. It is found that the

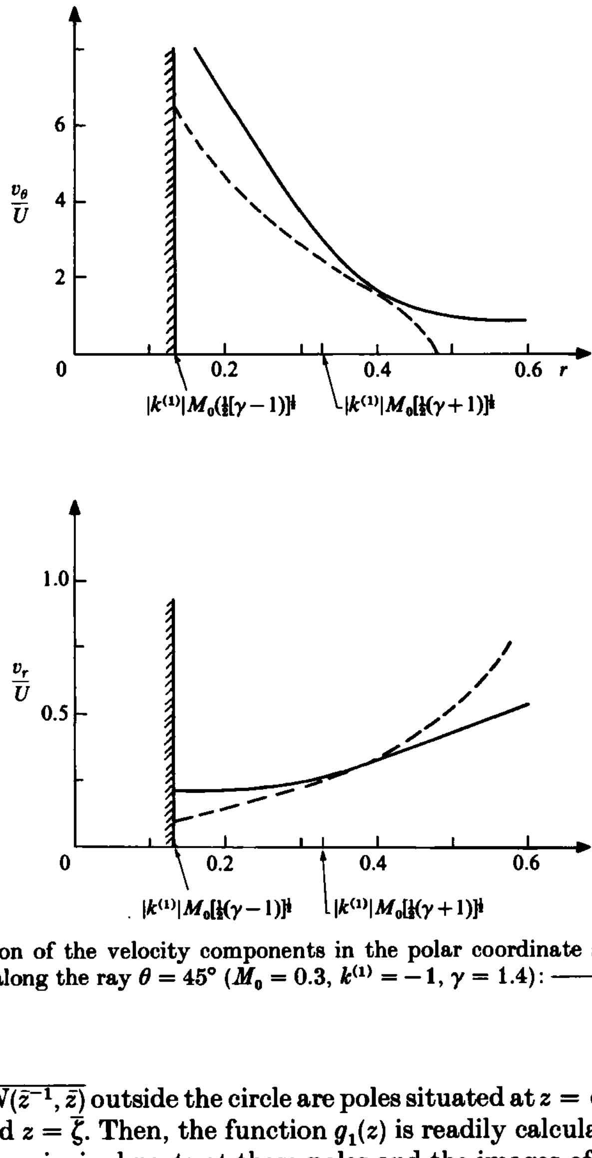

FIGURE 4. Variation of the velocity components in the polar coordinate system centred at the starboard vortex along the ray $\theta = 45°$ ($M_0 = 0.3$, $k^{(1)} = -1$, $\gamma = 1.4$): ——, outer solution; - - -, inner solution.

singularities of $\overline{N(\bar{z}^{-1}, \bar{z})}$ outside the circle are poles situated at $z = \infty$ and at the vortex centres $z = \zeta$ and $z = \bar{\zeta}$. Then, the function $g_1(z)$ is readily calculated by subtracting from $\overline{N(\bar{z}^{-1}, \bar{z})}$ its principal parts at these poles and the images of the principal parts with respect to the circle $|z| = 1$. By this procedure, the function $g(z)$ and, consequently, $f_1(z, \bar{z})$, is obtained in closed form. The result is quite lengthy and is given explicitly by Yungster (1985) along with the expressions for the coordinates of the vortices. The inner solutions near the centres of the two vortices are obtained by introducing into (31) the values of k and C_1 for each vortex.

Figure 2 shows the stationary position of the starboard vortex as a function of the parameter M_0 and the non-dimensional strength $k^{(1)}$ of the vortex. It is seen that the effects of compressibility on the computed equilibrium position of the vortex become stronger as M_0 and $|k^{(1)}|$ increase, and their trend is to move the vortex closer to both the axis of symmetry and the circle. We note, however, that when M_0 and $|k^{(1)}|$ exceed some critical combination of their values, the present approximation breaks down, since the velocities at the circle become sonic or the regions of supersonic flow around the vortex centres intersect. The positions of the vortex for these critical values are

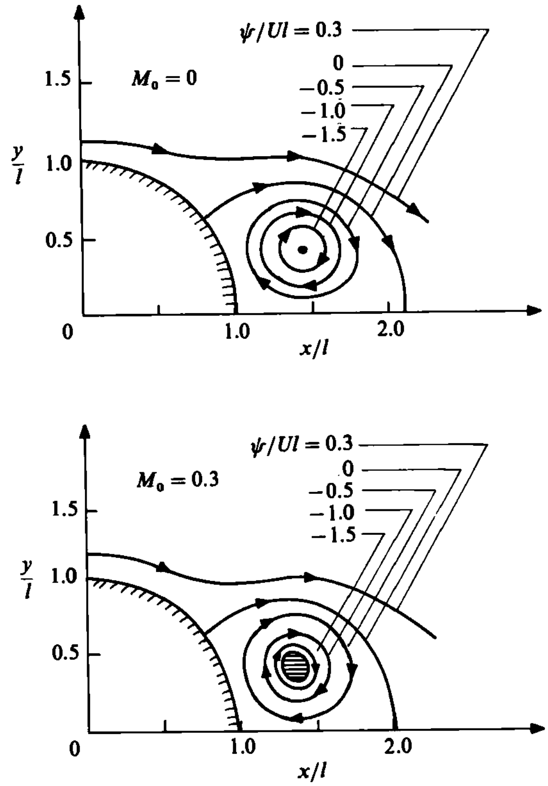

FIGURE 5. Streamline pattern at the lee-side of the circle ($k^{(1)} = -0.8$).

shown in figure 2 by the broken line and the validity of our results is restricted to the vortices situated below this line.

The computed equilibrium positions are compared in figure 2 with experimental data measured at the cylindrical after-part of ogive-cylinder bodies at high angles of attack (Fidler, Nielson & Schwind 1977; Jorgensen 1977; Owen & Johnson 1979). The measured vortex positions – both in the incompressible and compressible cases – are in close proximity to the computed curves for the corresponding M_0, whereas the measured vortex strengths $|k_m|$ are less than predicted. Consequently, for the same strengths, the computed vortex positions are situated closer to the body than those measured. This displacement is consistent with the omission of three-dimensional effects in the present model. In the compressible case, the measured vortex positions are placed outside the validity boundary of the present theory (they correspond to the appearance of supersonic velocities at the x-axis). However, the consistency of the results in this case can be explained, since the velocities along the x-axis become subsonic when the measured vortex strengths are used in the computation.

Figure 3 shows the velocity at the circle as function of the arc β measured from the forward stagnation point. The effects of compressibility are found to be strong in the vicinity of $\beta = \pm 90°$ and at the rear part of the circle that is situated close to the vortices. The matching of the outer and the inner solutions is illustrated in figure 4. It gives the variations along the ray $\theta = 45°$ of the velocity components v_r and v_θ in the polar coordinate system centred at the vortex. In the calculations we chose $\gamma = 1.4$. The results corresponding to the two solutions nearly coincide at values of r slightly larger than $|k^{(1)}M_0|\,[\frac{1}{2}(\gamma+1)]^{\frac{1}{2}}$, indicating that the matching is achieved at high subsonic velocities.

A comparison between the streamline patterns of the incompressible and the compressible flow at the lee-side of the circle is shown in figure 5. We found that outside the region shown, the compressibility effects on the streamlines are negligible in the range of applicability of the present solution, i.e. for $0 \leqslant M_0 \leqslant 0.35$.

5. Conclusions

A method has been developed to calculate compressibility corrections for the potential, steady and plane flow past a contour and stationary vortices. The method involves the asymptotic matching of an outer and an inner solution and the determination of the stationary vortex positions. The outer solution is valid throughout the flow field except in the vicinities of the vortex centres. Its calculation was reduced to the evaluation of the complex velocity potential of the incompressible flow past the configuration considered and of an additional analytic and bounded complex function. The inner solution, which governs the flow near the centre of any stationary vortex surrounded completely by subsonic flow, was obtained in closed form. In essence, the whole method can be regarded as the extension of the second approximation of Imai–Lamla to the case when there are vortices in the flow. It holds as long as the free stream and the velocities at the contour are subsonic and the regions of supersonic flow around the vortices do not intersect.

The method was applied to the flow past a circle and a symmetric pair of stationary vortices in a subsonic stream. The results show that the effects of compressibility on the equilibrium positions of the vortices and on the velocities can be significant. The compressibility corrections are qualitatively consistent with experimental results. The effects of compressibility on the streamlines are noticeable only in a bounded region that includes the vortices.

We note that the procedure outlined in §2 for the calculation of the compressibility corrections to the vortex position has also been applied to the flow past a Joukowski airfoil with a trapped vortex (Barsony-Nagy 1985).

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